Deriving the Linear-Wave Spectrum from a Nonlinear Spectrum

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Abstract: In extreme nonlinear seas, one cannot directly use the measured spectra, $S_T(\omega)$, from these seas in an analysis, or to derive a seakeeping prediction, but rather one must derive the underlying linear spectrum to describe the waves that should be simulated. At extreme wave heights theoretical spectra have nonlinear tails that are unrealizable in an experimental facility due to the breaking of high frequency waves. A technique for deriving the underlying realizable spectrum is described.

Keywords: Nonlinear spectrum, second-order spectrum, linear spectrum, linear spectrum from nonlinear spectrum

1 Introduction

In extreme nonlinear seas, one cannot directly use the measured spectra, $S_T(\omega)$, from these seas in an analysis, or to derive a seakeeping prediction, but rather one must derive the underlying linear spectrum to describe the waves that should be simulated. This is because nonlinear interactions between the linear waves will provide second-order, nonlinear contributions through the physics capturing wave-wave interactions.

At extreme wave heights theoretical spectra such as the Joint North Sea Wave Observation Project (JONSWAP) spectrum have nonlinear tails that are unrealizable in an experimental facility due to the breaking of high frequency waves. The underlying realizable spectrum may be derived as the corresponding linear spectrum by the tech- niques to be described.

The derivation of the linear spectrum underlying the nonlinear spectrum requires the solution of an integral equation describing the measured spectrum by either direct or indirect methods. This section will introduce two possible methods of solving this problem, with the assumption that the process involves only first- and second-order processes, a reasonable assumption in most circumstances.

2 Determining the Linear Spectrum

Wave processes are assumed to be homogeneous, stationary, and ergodic. This allows us to derive all statistical properties of the wave height and the power spectrum by examining wave records at just one position, which is taken as x = 0.

Only the case of unidirectional waves is considered here since an integral equation similar to the one that exists for unidirectional waves is not known for the case of multidirectional waves. A two-sided target spectrum $S_Q(\omega)$ is assumed to have been provided by the user. A two-sided linear spectrum $S_L(\omega)$ is sought which approximately satisfies the equation

$$S_Q(\omega) = S_L(\omega) + 2\int_{-\infty}^{\infty} d\sigma\, S_L(\sigma)\, S_L(\omega-\sigma) Z^2(\sigma,\omega-\sigma) \tag{1}$$

for real ω where

$$Z(\sigma,\omega) = \begin{cases} (\sigma^2+\omega^2)/(2g) & \text{if } \omega\sigma > 0 \\ -|\sigma^2-\omega^2|/(2g) & \text{if } \omega\sigma < 0 \end{cases}.$$

The details of the derivation are presented in Sclavounos (1992). The spectral density $S_L(\omega)$ is that of the linear model and is defined as follows:

$$\frac{1}{8}a_j^2 = S_L(\omega_j)\Delta\omega\,. \tag{2}$$

Therefore, the statistical inference of the second-order model reduces to the determination of the wave amplitudes a_j so that the second-order spectral density best matches the measured spectrum $S_Q(\omega)$. The linear spectral density $S_L(\omega)$ may be selected from any of the standard families with parameters such that the equality (1) is satisfied in a least squares sense.

For example, the ITTC spectrum may be used for the representation[1] of $S_L(\omega)$:

$$S_L(\omega) = \frac{0.110}{4\pi} H_{1/3}^2 T_1 \lambda^{-5} e^{-0.440\lambda^{-4}}, \quad \lambda = \frac{\omega T_1}{2\pi}. \tag{3}$$

In (3) an accurate estimate of the modal period T_1 may be available from full-scale measurements. The significant wave height on the other hand must be selected so that (1) is satisfied as accurately as possible, given $S_Q(\omega)$. The amplitudes of the regular wave components then follow from (2).

A numerical approach such as the following might be considered. Using this definition of Z and assuming that the spectra $S_L(\omega)$ and $S_Q(\omega)$ are even functions of ω, the integral equation can be rewritten as

$$S_Q(\omega) = S_L(\omega) + 2\int_0^\infty d\sigma\, S_L(\sigma) \times \left[S_L(\omega+\sigma)Z^2(-\sigma,\omega+\sigma) + S_L(\omega-\sigma)Z^2(\sigma,\omega-\sigma)\right].$$

The integral equation has no solution if the target spectrum has content of higher than second order in the wave amplitude. This section describes how a least-squares approximation to the desired linear spectrum $S_L(\omega)$ may be obtained and thus avoids the issue of whether a solution exists or not.

The numerical scheme that follows requires that discrete frequencies be equally spaced. If this is not the case, then $\omega - \sigma$ in the discretized integral equation will not be one of the discrete frequencies ω_j and any numerical scheme becomes complicated. The discrete frequencies in this section are therefore not necessarily those for which linear wave amplitudes a_j are chosen in the next section, and the N used in the description of the numerical scheme is not necessarily the number of positive wave frequencies used in the next section.

To discretize this equation, it is assumed that $S(\omega)$ can be ignored for $|\omega| > \Omega$. Then N and $\Delta\omega$ are chosen so that $N\Delta\omega = \Omega$. Define the following quantities for $j = -N, -N+1, \ldots, N$:

$$\omega_j = j\Delta\omega$$
$$\sigma_j = j\Delta\omega$$
$$S_j = S(j\Delta\omega)$$

We need to define $S_{L,j}$ over the wider range of j values between $-2N$ and $2N$:

$$S_{L,j} = S_L(j\Delta\omega)$$

although it has been assumed that the target spectrum $S_T(\omega)$ is negligible for values of j greater than N in absolute value. (The linear part of it in $S_L(\omega)$ should be even more negligible.) The range of integration is truncated and trapezoidal quadrature is used to approximate the integral equation as follows:

$$\begin{aligned} S_Q(\omega) &\approx S_L(\omega) + 2\int_{-\Omega}^{\Omega} d\sigma\, S_L(\sigma)S_L(\omega-\sigma)|Z(\sigma,\omega-\sigma)|^2 \\ &\approx S_L(\omega) + 2\Delta\sigma \Big\{ \frac{1}{2} S_L(\sigma_{-N})S_L(\omega-\sigma_{-N}) \times |Z(\sigma_{-N},\omega-\sigma_{-N})|^2 \\ &\quad + \sum_{n=-N+1}^{N-1} S_L(\sigma_n)S_L(\omega-\sigma_n) \times |Z(\sigma_n,\omega-\sigma_n)|^2 \\ &\quad + \frac{1}{2} S_L(\sigma_N)S_L(\omega-\sigma_N) \times |Z(\sigma_N,\omega-\sigma_N)|^2 \Big\} \end{aligned}$$

$$\begin{aligned} S_{Q,\ell} = S_{L,\ell} + 2\Delta\sigma \Big\{ &\frac{1}{2} S_{L,-N}S_{L,\ell+N} \times |Z(-N\Delta\omega,(\ell+N)\Delta\omega)|^2 \\ &+ \sum_{n=-N+1}^{N-1} S_{L,n}S_{L,\ell-n}\,|Z(n\Delta\omega,(\ell-n)\Delta\omega)|^2 \\ &+ \frac{1}{2} S_{L,N}S_{L,\ell-N}\,|Z(N\Delta\omega,(\ell-N)\Delta\omega)|^2 \Big\} \end{aligned}$$

Here $\Delta\sigma = \Delta\omega$. Taking account of where the target spectrum and its linear part are neglibigle, we have the following approximation of the integral equation:

$$S_{Q,\ell} \approx S_{L,\ell} + 2\Delta\omega \sum_{n=-N}^{N} S_{L,n}S_{L,\ell-n}Z_{n,\ell-n}^2 \tag{4}$$

[1]This representation can be obtained from equations on page 38 of Beck et al. (1989) if three significant digits are retained.

where $Z_{n,\ell-n} = Z(n\Delta\omega, (\ell-n)\Delta\omega)$.

A small error has been introduced at the two end points. When $|\ell-n| > N$, we can either ignore $S_{n-\ell}$ or set it equal to S_ℓ. The latter alternative is preferable. In view of the definition of the function Z and the evenness of S_L, (4) can be written as

$$S_{Q,\ell} \approx \begin{cases} S_{L,\ell} + 2\Delta\omega \sum_{n=1}^{N} S_{L,n} S_{L,|\ell-n|} Z^2_{n,\ell-n} & \text{if } \ell > 0 \\ S_{L,\ell} + 2\Delta\omega \sum_{n=-N}^{-1} S_{L,n} S_{L,-|\ell-n|} \times Z^2_{n,-\ell-n} & \text{if } \ell < 0 \end{cases} \tag{4'}$$

where the case $\ell = 0$ has been ignored since spectra are assumed to vanish as the frequency approaches zero.

Here $S_{L,p} = S_L(p\Delta\omega)$ and $S_{Q,p} = S_Q(p\Delta\omega)$. The series is truncated and the equations are written as

$$f_\ell \equiv S_{L,\ell} - S_{Q,\ell} + 2\Delta\omega \left[\sum_{n=1}^{\ell-1} S_{L,n} S_{L,\ell-n} Z^2_{n,\ell-n} + \sum_{n=\ell+1}^{N} S_{L,n} S_{L,n-\ell} Z^2_{n,\ell-n} + \sum_{n=1}^{N-\ell} S_{L,n} S_{L,n+\ell} Z^2_{-n,\ell+n} \right] = 0$$

for $\ell = 1, 2, \ldots, N$. The frequency $\Delta\omega$ and the number N are provided by the user. The objective is to minimize the sum

$$\chi^2 = \sum_{\ell=1}^{N} f_\ell^2 .$$

An initial guess $S^{(0)}_{L,\ell}$ for the discrete linear spectrum is provided by the equation

$$S^{(0)}_{L,\ell} = S_{T,\ell} \quad \text{for } \ell = 0, 1, \ldots, N .$$

All iterates for the linear spectrum are assumed to vanish at $\omega = 0$ rad/sec:

$$S^{(p)}_{L,0} = 0 \quad \text{for } p = 0, 1, \ldots$$

It is now assumed that the p-th iterate, say $S^{(p)}_{L,m}$, is known. For $m = 1, 2, \ldots, N$, $S^{(p+1)}_{L,m}$ is chosen between $(1-\alpha)\, S^{(p)}_{L,m}$ and $(1+\alpha)\, S^{(p)}_{L,m}$ such that

$$\sum_{\ell=0}^{N} f_\ell^2 \left(S^{(p+1)}_{L,1}, \ldots, S^{(p+1)}_{L,m-1}, S^{(p+1)}_{L,m}, S^{(p)}_{L,m+1}, \ldots, S^{(p)}_{L,N} \right) .$$

is approximately minimized. The number α is somewhat arbitrary and can be provided by the user; it only serves to bound the interval in which a minimum of χ^2 is sought. Numerical tests for some spectra indicate that $\alpha = 0.1$ is acceptable for those spectra. To minimize χ^2, we can check the sum at several, say 10, evenly spaced points $S^{(p+1)}_{L,m}$ in the interval $[(1-\alpha)\, S^{(p)}_{L,m}, (1+\alpha)\, S^{(p)}_{L,m}]$ and make the change based on the 10 evaluations of χ^2. The number 10 is arbitrary and can be replaced by another value supplied by the user. Furthermore, the points do not have to be evenly spaced. The whole process is repeated for a specified number of iterations. The sum χ^2 can be monitored and the iterative process can be truncated when the fractional change in the sum is less than a user-specified tolerance or no longer decreases.

The desired values $S_{L,\ell}$ for the discrete linear spectrum are given by $S^{(p)}_{L,m}$ where p is the number of the most recent iterate. Interpolation is required if the spectral density function is desired at frequencies other than $\omega_m = m\Delta\omega$.

3 The Algorithm

The scheme proceeds as follows:

1. Choose the highest frequency Ω to be used in the discretization and a sufficiently large value of N. This defines $\Delta\omega = \Omega/N$.
2. Define the initial guess $S^{(0)}_{L,\ell}$ as the target spectrum: $S^{(0)}_{L,\ell} = S_{T,\ell}$. Also define $S^{(0)}_{Q,\ell}$ so that $S^{(0)}_{Q,\ell} = S^{(0)}_{L,\ell}$. Except for the zeroth iterate, the discrete function $S^{(m)}_{Q,\ell}$ is the right side of eq. (4) when $S_{L,\ell}$ is replaced by the m-th iterate $S^{(m)}_{L,\ell}$ for linear part of the target spectrum.
3. This is the start of a loop. Calculate a new guess $S^{(m)}_{L,\ell}$ for the linear part of the target spectrum from the equation

$$S^{(m)}_{L,\ell} = S_{T,\ell} \frac{S^{(m-1)}_{L,\ell}}{10^{-20} + S^{(m-1)}_{Q,\ell}} \tag{5}$$

4. Calculate the RMS difference between $S_{L,\ell}^{(m)}$ and $S_{L,\ell}^{(m-1)}$. If it is sufficiently small or there have been too many iterations, exit the loop.
5. Calculate the right side $S_{Q,\ell}^{(m)}$ of eq. (4) corresponding to $S_{L,\ell}^{(m)}$.
6. Go back to the beginning of the loop.

4 Numerical Test

The scheme was tested on the Bretschneider spectrum. Two tests were conducted, in the first, a quadratic target spectrum was constructed from the Bretschneider spectrum and the Bretschneider spectrum was extracted from this target spectrum. In the second test, the Bretschneider spectrum was treated and the target quadratic spectrum and the linear part of the Bretschneider was determined. In both cases, the Bretschneider spectrum was given by:

$$S(\omega) = A/\omega^5 e^{-B/\omega^4},$$

where $A = 173 H_{1/3}^2/T_1^4$, $B = 691/T_1^4$, $T_1 = 0.773\,T_m$, $T_m = 2\pi/0.45$ sec and $H_{1/3} = 14.7$ m.

In these computations, the loop was not terminated for any value of the RMS difference between iterates. Instead, the loop was allowed to execute 30 times. The parameter Ω was set to 10 rad/sec and the number of subintervals into which the interval $[0,\Omega]$ was subdivided was set to 1023 so that $N = 1024$.

For the first test case, where a target quadratic spectrum was constructed from a Bretschneider spectrum, Fig. 1 compares the initial iterate (the target spectrum $S_T = S_L^{(0)}$) with the 10th, 20th, and 30th iterates for $S_L^{(m)}$. Figure 2 depicts the nonlinear part $S_T - S_L^{(m)}$ of the computed total spectrum at the mth iterate for for $m = 10, 20, 30$ and the target Bretschneider spectrum.

For the second test case, where the Bretschneider spectrum is considered to be the target spectrum, Fig. 3 compares the initial iterate (the target spectrum $S_T = S_L^{(0)}$) with the 10th, 20th, and 30th iterates for $S_L^{(m)}$. Figure 4 plots the computed linear spectrum $S_L^{(m)}$ and the computed total spectrum $S_Q^{(m)}$ at iteration $m = 30$. Figure 5 is a log plot of the RMS difference between successive iterates $S_L^{(m)}$ for the linear part of the target spectrum. Even though successive approximations $S_L^{(m)}$ appear to converge, Figs. 3 and 4 show

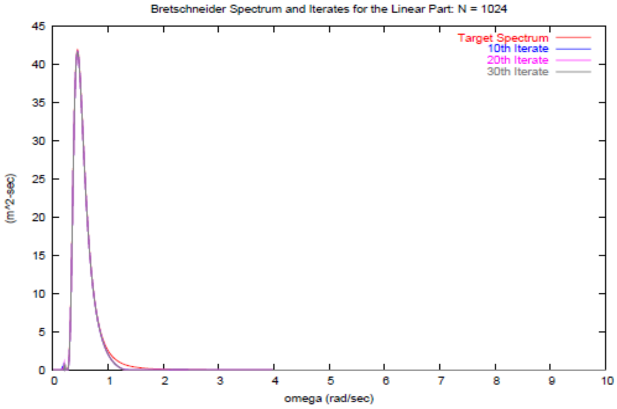

Fig. 1 The target spectrum and iterates for the linear part of the target spectrum. Spectral definition: $S(\omega) = A/\omega^5 e^{-B/\omega^4}$, $A = 173 H_{1/3}^2/T_1^4$, $B = 691/T_1^4$, $T_1 = 0.773\,T_m$, $T_m = 2\pi/0.45$ sec and $H_{1/3} = 14.7$ m.

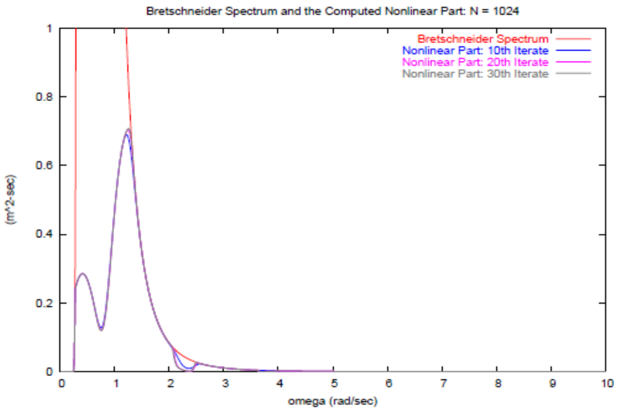

Fig. 2 The nonlinear part of the target spectrum calculated as the difference between the target Bretschneider spectrum and iterates for the linear part of the target spectrum. Spectral definition: $S(\omega) = A/\omega^5 e^{-B/\omega^4}$, $A = 173 H_{1/3}^2/T_1^4$, $B = 691/T_1^4$, $T_1 = 0.773\,T_m$, $T_m = 2\pi/0.45$ sec and $H_{1/3} = 14.7$ m.

that the result for the linear part of the Bretschneider spectrum is not the limit of the successive approximations $S_{L\ell}^{(m)}$:

$$\lim_{m\to\infty} S_{L\ell}^{(m)} \neq S_{L\ell}.$$

The computed nonlinear spectrum is larger than the target Bretschneider spectrum in finite intervals, a phenomena which probably also occurs at higher frequencies. As an example, near the frequency $\omega \approx 7.5$ rad/sec in Figs. 3 and 4, it is clear that the computed nonlinear spectrum is larger than the target spectrum and that the linear part of it is identically zero (within a small tolerance). In this method a local change in the linear spectrum does not affect the computed nonlinear spectrum in the neighborhood of the change as much as at other frequencies due to the nonlinear nature of the integrand in the integral equation. In any case, the linear spectrum cannot be reduced below its current value near $\omega = 7.5$ rad/sec so that a beneficial local change in the linear spectrum is precluded. Thus, it is impossible to improve the solution when the next iterate is computed from (5).

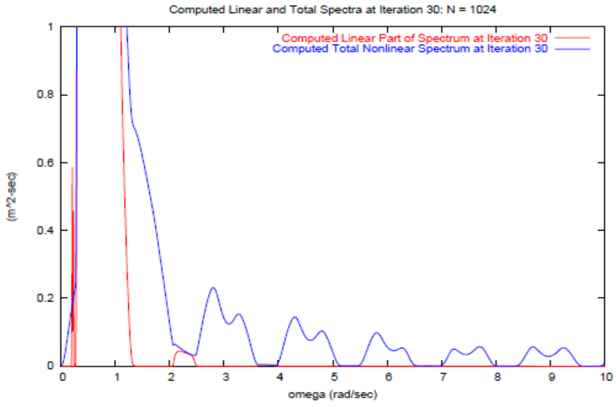

Fig. 4 Comparison of the total computed nonlinear spectrum with the linear part of the total computed nonlinear spectrum. Spectral definition: $S(\omega) = A/\omega^5 e^{-B/\omega^4}$, $A = 173 H_{1/3}^2/T_1^4$, $B = 691/T_1^4$, $T_1 = 0.773\, T_m$, $T_m = 2\pi/0.45$ sec and $H_{1/3} = 14.7$ m.

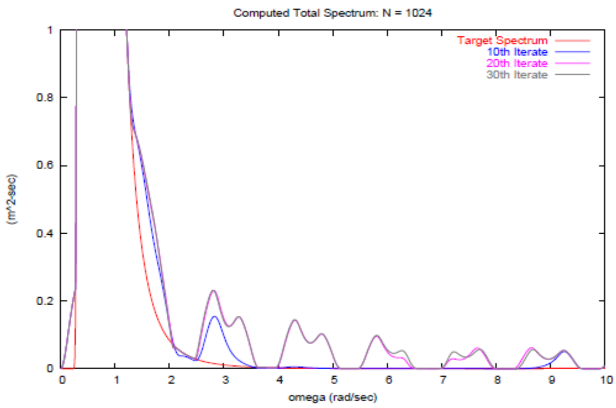

Fig. 3 The total computed nonlinear spectrum obtained while iterating for for the linear part of the Bretschneider spectrum. Spectral definition: $S(\omega) = A/\omega^5 e^{-B/\omega^4}$, $A = 173 H_{1/3}^2/T_1^4$, $B = 691/T_1^4$, $T_1 = 0.773\, T_m$, $T_m = 2\pi/0.45$ sec and $H_{1/3} = 14.7$ m.

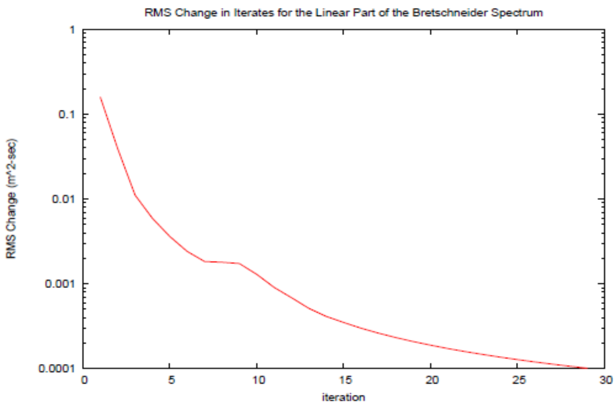

Fig. 5 Log10 plot of the RMS difference between successive iterates for the linear part of the target Bretschneider spectrum. Spectral definition: $S(\omega) = A/\omega^5 e^{-B/\omega^4}$, $A = 173 H_{1/3}^2/T_1^4$, $B = 691/T_1^4$, $T_1 = 0.773\, T_m$, $T_m = 2\pi/0.45$ sec and $H_{1/3} = 14.7$ m.

5 Conclusions

In extreme nonlinear seas, one cannot directly use the measured spectra, $S_T(\omega)$, from these seas in an analysis, or to derive a seakeeping prediction, but rather one must derive the underlying linear spectrum to describe the waves that should be simulated. A technique for deriving the underlying realizable spectrum has been described.

Using the technique presented for deriving the linear spectrum from the quadratic spectrum via the solution of an integral equation, even though successive approximations $S_L^{(m)}$ appear to converge, the result for the linear part of the spectrum is not the limit of the successive approximations. It is clear that in some intervals, the computed nonlinear spectrum is larger than the target spectrum and that the linear part of it is identically zero (within a small tolerance). In this method a local change in the linear spectrum does not affect the computed nonlinear spectrum in the neighborhood of the change as much as at several other frequencies due to the nonlinear nature of the integrand in the integral equation.

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